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THE CHEAPEST HEDGE: A PORTFOLIO DOMINANCE APPROACH

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The cheapest hedge:
A portfolio dominance approach*

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ABSTRACT. Investors often wish to insure themselves against the payoff of their portfolios falling below a certain value. One way of doing this is by purchasing an appropriate collection of traded securities. However, when the derivatives market is not complete, an investor who seeks portfolio insurance will also be interested in the cheapest hedge that is marketed. Such insurance will not exactly replicate the desired insured-payoff, but it is the cheapest that can be achieved using the market.

Analytically, the problem of finding a cheapest insuring portfolio is a linear programming problem. The present paper provides an alternative *portfolio dominance* approach to solving the minimum-premium insurance portfolio problem. This affords remarkably rich and intuitive insights to determining and describing the minimum-premium insurance portfolios.

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1. Introduction

Portfolio insurance guarantees a minimum payoff or floor on the downside while capturing the upside. The desired insured payoff can be replicated by holding a riskless asset and fiduciary call options. Alternatively, it can be replicated by holding the portfolio and protective put options.

When derivative markets are not complete, the desired insured payoff need not be marketed and a perfectly insuring portfolio may not be available. However, there always exist tradable portfolios that pay more in every state than the desired payoff. These portfolios are candidates for portfolio insurance when markets are not complete. The price of such a *super insuring* portfolio is its insurance-premium. Therefore, an investor who seeks portfolio insurance would be interested in the cheapest hedge that combines available securities, even though it need not exactly replicate the desired insured payoff. That is, an investor will strive to purchase a portfolio whose payoff dominates the desired insured payoff and which has the lowest insurance-premium. Such a portfolio is termed a *minimum-premium insurance portfolio.*

The problem of finding a minimum-premium insurance portfolio is a standard linear programming problem. This paper presents an alternative approach to solving the minimum-premium insurance portfolio problem in a general setting. This is done by taking advantage of the order theoretic structure of portfolio dominance—whereby a portfolio dominates another portfolio if it pays at least as much in each state of the world. As we shall see, the portfolio dominance approach affords remarkably rich and intuitive insights to determining and describing minimum-premium insurance portfolios.

The principal insight of this paper is that we can always obtain a minimum-premium insurance portfolio by looking at portfolio dominance over a restricted number of states of the world. In particular, its analysis focuses on the structure of portfolio dominance over as many uncertain states as available securities.

Technically, the argument goes as follows. When markets are complete, it is easy to determine the portfolio that replicates a desired insured payoff, since in such a setting there are as many states of the world as the available (non-redundant) securities. In terms of portfolio dominance, this portfolio is the least upper bound of the underlying portfolio and the floor.[1]

In contrast, when markets are not complete there are more states in the world than available securities and the desired insured payoff need not be marketed. In such a case, we construct a number of different notions of portfolio dominance by

[1]Here the matrix of non-redundant contingent claims is non-singular and the replicating portfolio can be calculated by taking the inverse value of the desired insured-payoff.

discarding enough states of the world. For instance, if there are J securities, then we can say that a portfolio dominates another portfolio if it pays at least as much in the first J states of the world. Likewise, a portfolio dominates another portfolio if it pays at least as much in the last J states of the world. Now for every such restricted notion of portfolio dominance, we can calculate the least upper bound of the underlying portfolio and the floor giving us a finite number of candidate portfolios. The main result of this paper asserts the following:

- *One of the finite number of the least upper bounds or candidate portfolios of the underlying portfolio and the floor must be a minimum-premium insurance portfolio.*

A characterization of investors that demand portfolio insurance has been presented in the classical article of H. Leland [11]; where it is assumed that option markets are complete and therefore any desired insured payoff can be perfectly replicated through the purchase of traded securities. Clearly, an investor that demands insurance in a complete market also demands insurance in the case of an incomplete derivatives market. However, if she cannot perfectly replicate the desired insured payoff, then why would she be interested in the "exotic" insurance studied in the present paper? Why characterize the cheapest hedge? Why the minimum cost criterion?

The replication of derivatives in constrained markets using the minimum cost criterion has been the subject of many articles in the literature. For instance, V. Naik and R. Uppal [13] use the minimum cost criterion to construct optimal hedging strategies in the presence of leverage constraints. Their work determines the strategy that minimizes the initial cost of hedging given leverage constraints. They argue that the criterion of minimum cost has several advantages. First, for constrained institutions that need to hedge liabilities, this approach is equivalent to maximizing profit. Second, the minimum cost approach also determines the maximum price that a constrained investor would be willing to pay for a contingent claim for exact portfolio insurance. That is, it is the maximum price that an investor is willing to pay for a non-traded Over-The-Counter portfolio insurance. Third, the authors show how the minimum cost criterion is related to utility maximization in the presence of leverage constraints.

The minimum cost criterion is by now a well studied in the literature on hedging and option pricing under constraints. For example, C. Edirisinghe, V. Naik, and R. Uppal [5] study minimum-premium hedging in the presence of transactions costs. Moreover, M. Broadie, J. Cvitanic, and H. M. Soner [3] use the minimum cost criterion to determine the cheapest portfolio that dominates an option in the presence of extremely general constraints. Our analysis is motivated by the issues considered in these papers.

The cheapest hedge problems under portfolio constraints lend themselves comfortably to the realm of linear optimization (see for instance V. Naik and R. Uppal [13]) and to convex as well as to stochastic optimization approaches (see for instance C. Edirisinghe, V. Naik, and R. Uppal [5] and I. Karatzas and S. Kou [8, 9]). Why the portfolio dominance approach?

The portfolio dominance approach captures an important mathematical aspect of options—the building blocks of hedging strategies. Indeed, under the portfolio dominance approach an option is simply a vector lattice operation in the portfolio space. In fact, for an underlying security with replicating portfolio θ, the call option at strike price k is replicated by the portfolio $(\theta-\mathbf{k})^+$, where $\mathbf{k}$ is the riskless portfolio paying k in each state of the world and the lattice operation $(\theta-\mathbf{k})^+$ is taken in the space of portfolios. Similarly, the put option at strike price k is replicated by the portfolio $(\mathbf{k}-\theta)^+$. Furthermore, the portfolio dominance approach has already yielded several results on portfolio trading in complete as well as incomplete markets. See for instance the work of D. Brown and S. Ross [4] who extend Ross' classical result [14] on the role of options in completing markets (see also R. Green and R. Jarrow [6]).

The structure of this paper is as follows. The model is in Sections 2. The main result regarding the minimum-premium insurance portfolio is stated in Section 3. Section 4 illustrates the results with several examples. The mathematical background needed for establishing the main result is presented in Section A1 of the Appendix. Section A2 studies the concept of portfolio dominance, while the proof of the main result of this work is in Section A3.

2. Minimum-premium insurance portfolio

This section begins with a brief exposition of portfolio insurance in the standard state-space assets markets model, see for example the models in [14, 12]. We then look at hedging when markets are complete. Using the insights gained from the case of complete markets we extend the analysis to the case of incomplete markets.

We consider the two-period securities model. There is a finite number S of states of the world. Agents trade $J \leq S$ non-redundant securities $r_1, r_2, \ldots, r_J$ in period-zero whose period-one payoffs are state contingent claims. Therefore, we allow for incomplete markets in which the number of no-redundant securities J is smaller than the number of states S. As usual, the *asset returns matrix* (or the *payoff matrix*) R is the $S \times J$ matrix whose columns are the available no-redundant (i.e., linearly independent) security vectors:

$$R = \begin{bmatrix} r_1(1) & r_2(1) & \dots & r_J(1) \\ r_1(2) & r_2(2) & \dots & r_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ r_1(S) & r_2(S) & \dots & r_J(S) \end{bmatrix}$$

Portfolios are linear combinations of the available securities. A portfolio is therefore represented by a vector in $\mathbb{R}^J$. Portfolios are considered as column vectors and the payoff of a portfolio θ is $R\theta$.

A state contingent claim, which is a vector in $\mathbb{R}^S$, is said to be a *marketed payoff* if it lies in the asset span (i.e., the range) $M = \langle R \rangle$ of the returns matrix R in $\mathbb{R}^S$; in which case, there is a unique portfolio (called the *replicating portfolio*) of the available securities whose payoff is the state contingent claim. We shall assume that the riskless bond $\mathbf{1} = (1, 1, \dots, 1)$ is marketed.

We shall also say that a portfolio θ *super replicates* a state contingent claim $x \in \mathbb{R}^S$ if $R\theta \geq x$. That is, θ pays at least as much in each state as x. A portfolio θ *(perfectly) replicates* a state contingent claim $x \in \mathbb{R}^S$ over a set of states I if $R\theta(s) = x(s)$ for every $s \in I$.

If the asset span equals the whole space of contingent claims (i.e., if $J = S$), then markets are *complete*. When $J < S$ the markets are *incomplete* in which case some state contingent claims cannot be replicated by a portfolio.

We shall restrict our study to *arbitrage-free* security prices. That is, we restrict our attention to vectors $q \in \mathbb{R}^J$ of security prices that give a non-zero positive value $q \cdot \theta > 0$ to any non-zero portfolio θ with a positive payoff $R\theta \geq 0$. A price $q \in \mathbb{R}^J$ is *arbitrage free* (resp. *weakly arbitrage free*) if $q \cdot \theta > 0$ (resp. $q \cdot \theta \geq 0$) whenever the portfolio θ satisfies $R\theta > 0$ (resp. $R\theta \geq 0$).

Portfolio Insurance: The *insured payoff* of a portfolio $\theta = (\theta_1, \theta_2, \dots, \theta_J)$ at a *floor* $k \in \mathbb{R}$ is a state contingent claim that captures the upside of the portfolio and insures against any downside below the floor. In other words, the insured payoff is the state contingent claim

$$\max\{R\theta, \mathbf{k}\} = \begin{bmatrix} \max\left\{\sum_{j=1}^J r_j(1)\theta_j, k\right\} \\ \max\left\{\sum_{j=1}^J r_j(2)\theta_j, k\right\} \\ \vdots \\ \max\left\{\sum_{j=1}^J r_j(S)\theta_j, k\right\} \end{bmatrix},$$

where $\mathbf{k} = k\mathbf{1}$ is the riskless bond paying k in each state of the world. In a complete market the insured payoff of a portfolio is the contingent claim that can be replicated by holding the payoff of the portfolio and a put option with a strike price k; or it can be replicated by holding $\mathbf{k}$ and a call option on the portfolio with a strike price k. The basic problem is that when markets are incomplete the insured payoff need not be a marketed payoff.

Minimum–premium insurance portfolios: Once again we consider a portfolio θ and a floor k. Any portfolio η whose payoff $R\eta$ dominates the insured payoff $\max\{R\theta, \mathbf{k}\}$ in each state is viewed as an *insurance portfolio*. There are many such portfolios. The cost of such a portfolio is the *insurance-premium*. So, if q is a securities price, then the insurance-premium associated with an insurance portfolio η is $q \cdot \eta$. We are, therefore, interested in a *minimum–premium insurance portfolio* (or a *cheapest hedge portfolio*) of θ at the floor k, which is the least costly portfolio whose payoff dominates the insured payoff of θ and the floor k. That is, a minimum–premium insurance portfolio is a solution to the following minimization problem:

$$(MP) \quad \min q \cdot \eta$$
$$\text{s. t.: } \eta \in \mathbb{R}^J, \ R\eta \geq R\theta, \text{ and } R\eta \geq \mathbf{k}$$

A solution to this minimization problem always exists. As a matter of fact:

- *The solution set of the minimization problem (MP) is a non-empty, convex and compact subset of $\mathbb{R}^J$.*

3. Portfolio Dominance and the Cheapest Hedge Solution

In this section we shall sketch briefly the basic ideas behind our solution to the hedging problem. As mentioned before, our solution is based on the notion of portfolio dominance that is related to the lattice structures of the spaces.

We shall say that a portfolio θ *dominates* a portfolio η if $R\theta \geq R\eta$, in which case we write $\theta \succeq \eta$. The portfolio dominance relation $\succeq$ makes $\mathbb{R}^J$ a partially ordered vector space. We shall denote by C the (pointed convex) cone generated by $\succeq$, i.e,

$$C = \{\theta \in \mathbb{R}^J : \theta \succeq 0\}.$$

Now for any two portfolios θ and η we write $\theta \vee_C \eta$ to mean a least upper bound of the set $\{\theta, \eta\}$ relative to $\succeq$. That is, the portfolio $\theta \vee_C \eta$, if it exists, has the property that $\theta \vee_C \eta \succeq \theta$ and $\theta \vee_C \eta \succeq \eta$ and if $\mu \succeq \theta$ and $\mu \succeq \eta$, then $\mu \succeq \theta \vee_C \eta$.

Whenever markets are complete, one can calculate a unique portfolio that is a minimum–premium insurance portfolio for any arbitrage free securities price. However, when markets are not complete the minimum–premium insurance portfolio depends on the prevailing price. Nevertheless, as we shall see, the incomplete markets case is quite similar to the case of complete markets. The details follow.

Complete Markets: Assume for now that markets are complete. That is, assume that the payoff matrix R is a $J \times J$ matrix. Recall that we have fixed a portfolio θ and a floor k. When markets are complete, it is easy to calculate a perfect hedge, or a portfolio that replicates the insured payoff of θ at floor k.

Indeed, if the portfolio κ replicates $\mathbf{k}$ (i.e., if $R\kappa = \mathbf{k}$), then since R is invertible the insured payoff is replicated by the portfolio:

$$\theta^* = \theta \vee_C \kappa = R^{-1} \max\{R\theta, \mathbf{k}\}.$$

The portfolio θ^* is clearly a minimum–premium insurance portfolio for any arbitrage free price. In particular, it is independent of the prevailing arbitrage free security prices. That is, we have the following result.

Theorem 3.1. *If markets are complete, then for any arbitrage free price the unique minimum–premium insurance portfolio is replicated by the portfolio $\theta \vee_C \kappa$, which exists (and is the call option on the portfolio θ at strike price k and k bonds* $\mathbf{1}$.)

Incomplete Markets: Assume now that the market is incomplete. We shall see that discarding some $S - J$ states of the world allows us to use a procedure for calculating a minimum–premium portfolio insurance as though the market is complete. We shall describe this method next.

For any collection I of J elementary states let R_I be the $J \times J$ matrix whose rows are the rows of the payoff matrix R corresponding to the states of I. For instance, if there are three securities and four states then

$$R_{(1,3,4)} = \begin{bmatrix} r_1(1) & r_2(1) & r_3(1) \\ r_1(3) & r_2(3) & r_3(3) \\ r_1(4) & r_2(4) & r_3(4) \end{bmatrix}.$$

If R_I is invertible, then we say that R_I (or even I) defines a *pseudo-complete market*. Since the rank of R is J there always exists at least one pseudo-complete market.

Before proceeding further, let us introduce some further notation. If a set of states $I = \{s_1 < s_2 < \cdots < s_J\}$ defines a pseudo-complete market and θ is a portfolio, then we let $\theta_I = (\theta_{s_1}, \theta_{s_2}, \ldots, \theta_{s_J})$. If we view θ_I as a column vector, then we shall denote $R_I\theta_I$ by $R_I\theta$, that is, $R_I\theta = R_I\theta_I$.

Now each *pseudo-complete market* R_I generates a new notion of portfolio dominance $\succeq_I$ by defining $\theta \succeq_I \eta$ whenever $R_I\theta \geq R_I\eta$. It turns out that not only this portfolio dominance relation $\succeq_I$ partially orders the portfolio space $\mathbb{R}^J$ but it also induces a lattice ordering. That is, for every *pseudo-complete market* R_I its portfolio dominance cone

$$C_I = \{\theta \in \mathbb{R}^J : \ \theta \succeq_I 0\}$$

is a lattice cone—which is also a super-cone of C, i.e., $C \subseteq C_I$. This means that if η and θ are two portfolios, then the $\succeq_I$-supremum of the two portfolios $\theta \vee_I \eta$ exists and is given by $\theta \vee_I \eta = R_I^{-1} \max\{R_I\theta, R_I\eta\}$. Assuming that $R\kappa = \mathbf{k}$, for each pseudo-complete market R_I we let

$$\eta_I = \theta \vee_I \kappa = R_I^{-1} \max\{R_I\theta, \mathbf{k}\}.$$

If θ is any portfolio and k is a floor price, then a *potentially insuring portfolio* is any portfolio of the form η_I satisfying $R\eta_I \geq \max\{R\theta, \mathbf{k}\}$. We shall denote the finite collection of all potentially insuring portfolios of θ at the floor k by $\mathcal{P}_{\theta,k}$, i.e.,

$$\mathcal{P}_{\theta,k} = \left\{\eta \in \mathbb{R}^J : \ \eta = \eta_I \text{ for some pseudo-complete market } R_I \text{ and } R\eta \geq R\theta \vee \mathbf{k}\right\}.$$

Clearly, there is a finite number of potentially insuring portfolios that are calculated independently of the arbitrage free security price.

The remarkable property is that one of the potentially insuring portfolios is a minimum-insurance premium portfolio. This is the main result of this paper and it will be stated next. Its proof is quite involved and it will be presented in Section A3 of the Appendix.

Theorem 3.2 (The Cheapest Hedge Theorem). *For any portfolio θ, any arbitrage price q, and any floor k we have the following:*

(1) *There exists at least one potentially insuring portfolio $\theta \vee_I \kappa$ that is a minimum-premium insurance portfolio for θ at floor k.*
(2) *A minimum-premium insurance portfolio $\theta \vee_I \kappa$ is the cheapest potentially insuring portfolio. That is, $q \cdot (\theta \vee_I \kappa) \leq q \cdot \eta$ for all $\eta \in \mathcal{P}_{\theta,k}$.*
(3) *The portfolio $\eta^* = \theta \vee_C \kappa$ exists if and only if $\mathcal{P}_{\theta,k}$ consists of one portfolio η^*, which is automatically a minimum-premium insurance portfolio for any arbitrage free price.*

The third statement in the theorem is an extension of the main result in [1], which shows that a price independent minimum-premium insurance portfolio insurance exists for any portfolio–floor pair if and only if the portfolio dominance cone is a lattice cone; i.e., it generates vector lattice on the portfolio space.

Let us conclude this section with a final remark. There is an intuitively appealing way of identifying the potentially insuring portfolios:

- *A portfolio is a potentially insuring portfolio if and only if it super replicates the insured payoff and perfectly replicates the insured payoff over a set I of J states for which R_I is a pseudo-complete market.*

4. Illustrative examples

The Cheapest Hedge Theorem 3.2 can be reformulated as follows.

Theorem 4.1. *For any portfolio θ, any arbitrage free price q, and any floor k we have the following:*

1. *There exists at least one potentially insuring portfolio that is a minimum-premium insurance portfolio for θ at floor k.*
2. *A minimum-premium insurance portfolio can be obtained by solving the finite minimization problem:*

$$(\mathcal{F}MP) \qquad \min\ q\cdot\eta \qquad s.\,t.:\ \eta\in\mathcal{P}_{\theta,k}$$

3. *If $\mathcal{P}_{\theta,k}$ consists of one portfolio, say η^*, then η^* is automatically a minimum-premium insurance portfolio.*

That is: *For any arbitrage free price, the cheapest potentially insuring portfolio is a minimum-premium insurance portfolio.* In other words, we have reduced the minimum-premium insurance portfolio problem (MP) to the following minimization problem over a finite set.

$$\min\ q\cdot\eta \qquad \text{s. t.: } \eta \ \text{ is a potentially insuring portfolio.}$$

This section presents some illustrative examples of the preceding result. With this in mind, let θ be a portfolio, k a floor, and q an arbitrage free price. Moreover, for each set I of J states, let R_I be the $J\times J$ matrix whose rows are the rows of R determined by I. Now consider the following steps:

1. For each invertible R_I find the portfolio
$$\eta_I = R_I^{-1}\max\{R\theta,\mathbf{k}\}\,,$$
and form the collection $\mathcal{P}_{\theta,k}$ of all potentially insuring portfolios of θ at the floor k.
2. If $\mathcal{P}_{\theta,k}$ consists of one portfolio, say η^*, then we are done. The portfolio η^* is the only minimum-premium insurance portfolio for any arbitrage free price.
3. If $\mathcal{P}_{\theta,k}$ contains more than one portfolio, then the least costly portfolio η in $\mathcal{P}_{\theta,k}$ with respect to the price q is a minimum-premium insurance portfolio.

We are now ready to present three examples. The first example is an example of a complete market.

Example 1 (A Complete market). Suppose that there are four states of the world and that the market has the following non-redundant securities:

(1) A treasury bond with payoff $\mathbf{1} = (1,1,1,1)$.
(2) A corporate bond with payoff $(0,1,1,1)$.
(3) A share with payoff $(0,1,2,4)$.
(4) A call option on the share with a strike price of 3. That is, the security $\max\{(0,1,2,4) - \mathbf{3}, 0\} = (0,0,0,1)$.

Therefore, the asset returns matrix R is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 4 & 1 \end{bmatrix}.$$

Keep in mind that the payoff of any portfolio θ is $R\theta$.

Now consider the portfolio $\theta = (1,2,3,0)$. The insured payoff on a portfolio θ at a floor $k = 10$ is the contingent claim

$$\max\{R\theta, \mathbf{10}\} = \max\left\{ \begin{bmatrix} 1 \\ 6 \\ 9 \\ 15 \end{bmatrix}, \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 15 \end{bmatrix}.$$

This contingent claim is obviously marketed and is the payoff of the portfolio

$$\theta^* = R^{-1}\max\{R\theta, \mathbf{10}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Clearly, for any arbitrage free securities price q the portfolio θ^* is the unique minim-premium insurance portfolio. Figure 1 provides a graphical illustration of this example.

Example 2 (Incomplete markets with only one potentially insuring portfolio). We consider the market in the previous example. But now we suppose that the call option is not available. Thus, the market is described by the returns matrix

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

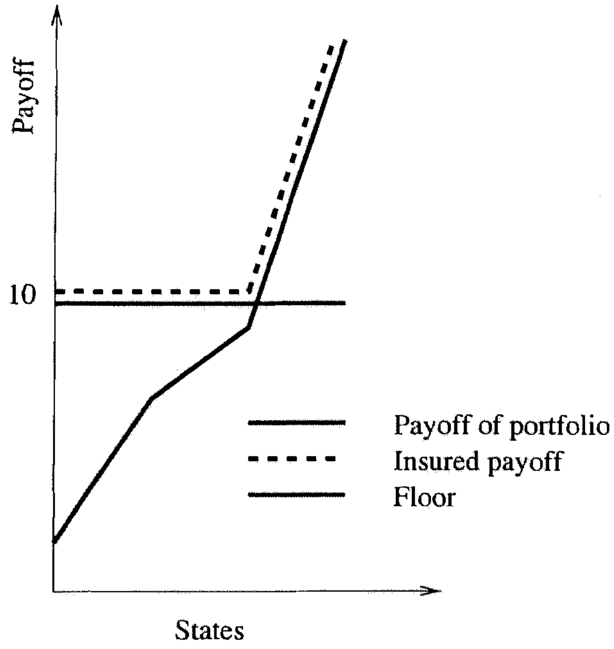

FIGURE 1. When markets are complete the insured payoff can be replicated by a portfolio containing ten treasury bonds and five call options.

Consider the portfolio $\theta = (1, 2, 3)$. The insured payoff on the portfolio θ at a floor $k = 10$ is once again the contingent claim

$$\max\{R\theta, \mathbf{10}\} = \max\left\{ \begin{bmatrix} 1 \\ 6 \\ 9 \\ 15 \end{bmatrix}, \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 15 \end{bmatrix}.$$

This contingent claim is not marketed since as we saw in the previous example it is the payoff of a portfolio using the unavailable call option.

However, we can calculate (at most) four important portfolios by looking at the four 3×3 matrices whose rows are taken from R. These are the matrices:

$$R_{(1,2,3)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad R_{(1,2,4)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 4 \end{bmatrix},$$

$$R_{(1,3,4)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}, \quad R_{(2,3,4)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Notice that $R_{(2,3,4)}$ is a singular matrix. Therefore, we restrict our attention to the remaining three pseudo-complete markets and obtain the following three portfolios:

$$\begin{aligned}
\eta_{(1,2,3)} &= R^{-1}_{(1,2,3)} \max\left\{R_{(1,2,3)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \\
\eta_{(1,2,4)} &= R^{-1}_{(1,2,4)} \max\left\{R_{(1,2,4)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 10 \\ -\frac{5}{3} \\ \frac{5}{3} \end{bmatrix}, \\
\eta_{(1,3,4)} &= R^{-1}_{(1,3,4)} \max\left\{R_{(1,3,4)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 10 \\ -5 \\ \frac{5}{2} \end{bmatrix}.
\end{aligned}$$

From these portfolios only $\eta_{(1,2,4)}$ has a payoff greater than the insured payoff of θ with floor 10. That is,

$$\theta^* = R\eta_{(1,2,4)} = \begin{bmatrix} 10 \\ 10 \\ \frac{35}{3} \\ 15 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 15 \end{bmatrix}.$$

Therefore, for any arbitrage free securities price q the portfolio $\eta_{(1,2,4)}$ is the only minimum-premium insurance portfolio. Therefore, we have found a solution that is independent of the arbitrage free security prices. This example is illustrated in Figure 2.

Example 3 (Incomplete markets with price dependent insurance). Consider a market with the payoff matrix

$$R = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 5 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix},$$

and, once again we consider the portfolio $\theta = (1, 2, 3)$.

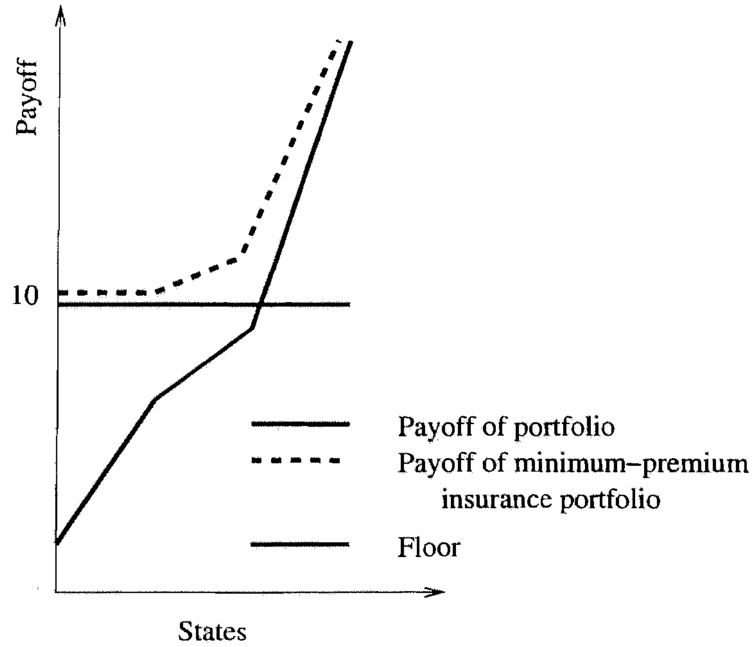

FIGURE 2. When the call option is not available the insured payoff cannot be replicated. However, the unique minimum-premium insurance portfolio contains ten treasury bonds, a short sale of one and two thirds of the corporate bond, and one and two thirds of the share.

The insured payoff on the portfolio θ at a floor $k = 10$ is the contingent claim

$$\max\left\{R\theta, \mathbf{10}\right\} = \max\left\{\begin{bmatrix}8\\18\\10\\1\end{bmatrix}, \begin{bmatrix}10\\10\\10\\10\end{bmatrix}\right\} = \begin{bmatrix}10\\18\\10\\10\end{bmatrix}.$$

This contingent claim is not marketed.

Next, we can calculate (at most) four portfolios by looking at the four 3×3 matrices whose rows are taken from R. These are the matrices:

$$R_{(1,2,3)} = \begin{bmatrix}1&2&1\\1&1&5\\1&0&3\end{bmatrix}, \quad R_{(1,2,4)} = \begin{bmatrix}1&2&1\\1&1&5\\1&0&0\end{bmatrix},$$

$$R_{(1,3,4)} = \begin{bmatrix}1&2&1\\1&0&3\\1&0&0\end{bmatrix}, \quad R_{(2,3,4)} = \begin{bmatrix}1&1&5\\1&0&3\\1&0&0\end{bmatrix}.$$

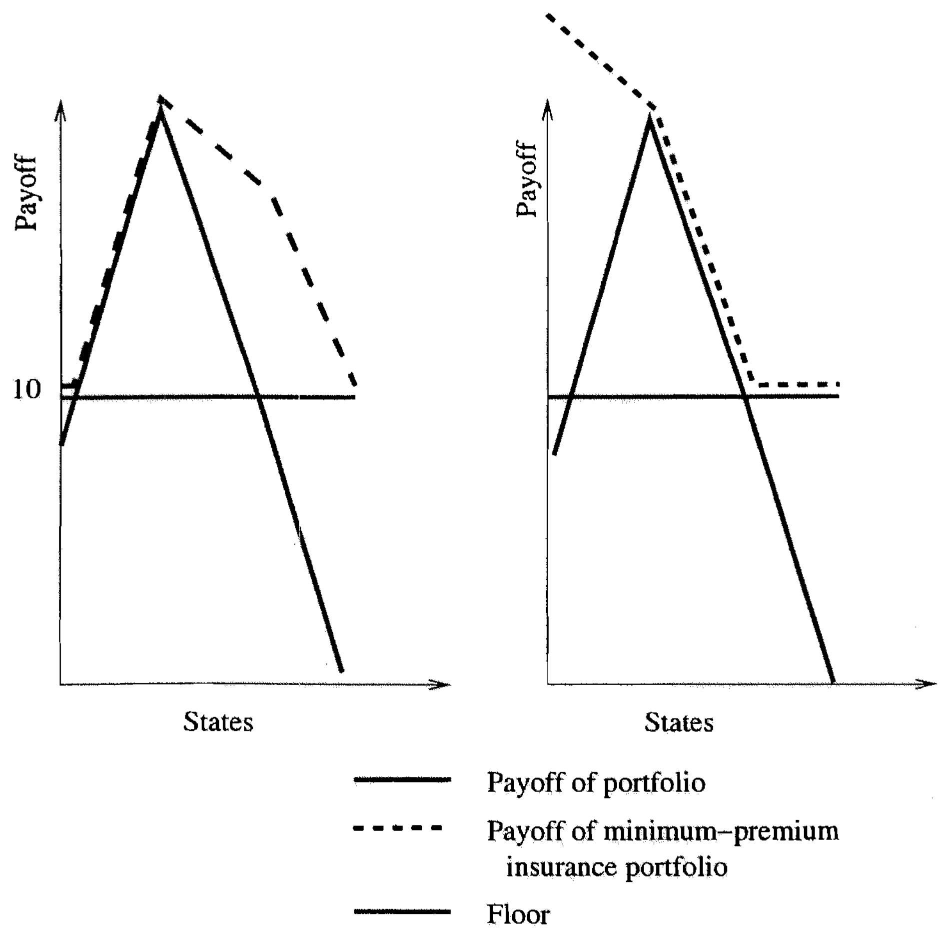

FIGURE 3. In this example the insured payoff which is a butterfly-spread cannot be replicated. However, there are two choices for portfolio insurance; and the choice depends on the prevailing securities prices.

All four matrices are invertible. So, we consider the portfolios:

$$
\begin{aligned}
\eta_{(1,2,3)} &= R^{-1}_{(1,2,3)} \max\left\{R_{(1,2,3)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 2 \\ \frac{8}{3} \\ \frac{8}{3} \end{bmatrix}, \\
\eta_{(1,2,4)} &= R^{-1}_{(1,2,4)} \max\left\{R_{(1,2,4)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 10 \\ -\frac{8}{9} \\ \frac{16}{9} \end{bmatrix}, \\
\eta_{(1,3,4)} &= R^{-1}_{(1,3,4)} \max\left\{R_{(1,3,4)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \\
\eta_{(2,3,4)} &= R^{-1}_{(2,3,4)} \max\left\{R_{(2,3,4)}\theta, \mathbf{10}\right\} = \begin{bmatrix} 10 \\ 8 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Notice that the portfolios $\eta_{(1,2,4)}$ and $\eta_{(2,3,4)}$ have a payoff greater than the insured payoff of θ at floor 10. (see Figure 3). That is,

$$R\eta_{(1,2,4)} = \begin{bmatrix} 10 \\ 18 \\ 15\frac{1}{3} \\ 10 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 18 \\ 10 \\ 10 \end{bmatrix} \quad \text{and} \quad R\eta_{(2,3,4)} = \begin{bmatrix} 26 \\ 18 \\ 10 \\ 10 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 18 \\ 10 \\ 10 \end{bmatrix} .$$

Now let us take three arbitrage free prices.

(1) Let $q = (1,1,1) = \frac{4}{9}(1,2,1) + \frac{1}{9}(1,1,5) + \frac{4}{9}(1,0,0)$. From

$$q \cdot \eta_{(1,2,4)} = 10\tfrac{8}{9} < q \cdot \eta_{(2,3,4)} = 18 \,,$$

we see that the minimum-premium insurance portfolio for the price q is $\eta_{(1,2,4)}$.

(2) For the arbitrage free securities price

$$q = (4,1,12) = \tfrac{1}{3}(1,2,1) + \tfrac{1}{3}(1,1,5) + \tfrac{10}{3}(1,0,3) \,,$$

we have

$$q \cdot \eta_{(1,2,4)} = 60 + \tfrac{4}{9} \quad \text{and} \quad q \cdot \eta_{(2,3,4)} = 48 \,.$$

Thus, $q \cdot \eta_{(1,2,4)} > q \cdot \eta_{(2,3,4)}$, and so $\eta_{(2,3,4)}$ is the minimum-premium insurance portfolio for the price $q = (4,1,12)$.

(3) For the price $q = (11,5,25) = 2(1,2,1) + 6(1,0,3) + 2(1,0,0) + (1,1,5)$, we get $q \cdot \eta_{(1,2,4)} = q \cdot \eta_{(2,3,4)} = 150$. Therefore, both portfolios are minimum-premium insurance portfolios for this price q.

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Appendix: Background and Proofs

A1. Mathematical preliminaries

We present here the basic concepts and results concerning cones in finite dimensional spaces that are needed to prove the main theorem of this paper. The generic finite dimensional vector space will be $\mathbb{R}^J$.

Recall that a *pointed convex cone*, or simply a *cone*, is a non-empty subset K of $\mathbb{R}^J$ such that:

(1) $K + K \subseteq K$,
(2) $\alpha K \subseteq K$ for each $\alpha \geq 0$, and
(3) $K \cap (-K) = \{0\}$.

Every cone K induces a vector space order $\geq_K$ (or $\leq_K$) on $\mathbb{R}^J$ by defining $x \geq_K y$ (or $y \leq_K x$) whenever $x - y \in K$. The vectors of K are precisely the vectors satisfying $x \geq_K 0$ and (if there is no other cone under consideration) they are referred to as *positive vectors*. We also write $x >_K 0$ to mean $x \geq_K 0$ and $x \neq 0$. For each vector $x \in K$, the K-order interval $\{y \in \mathbb{R}^J : \ 0 \leq_K y \leq_K x\}$ will be denoted $[0, x]_K$, i.e., $[0, x]_K = \{y \in \mathbb{R}^J : \ 0 \leq_K y \leq_K x\}$.

A cone K is said to be *generating* if $\mathbb{R}^J = K - K$, i.e., if every vector in $\mathbb{R}^J$ can be written as a difference of two vectors in K. The following result is well known and we state it for completeness.

Lemma A1.1. *A cone in $\mathbb{R}^J$ is generating if and only if it has an interior point.*

The *dual cone* K' of a cone K is defined by

$$K' = \left\{q \in (\mathbb{R}^J)' = \mathbb{R}^J : \ q \cdot x \geq 0 \ \text{ for all } \ x \in K\right\}.$$

The members of K' are called *positive linear functionals.*

Regarding dual cones, we have the following basic duality result.

Theorem A1.2 (Duality Theorem). *If K is a closed generating cone in $\mathbb{R}^J$, then:*

(1) *The dual cone K' is also a closed and generating cone.*
(2) *The dual cone of K' coincides with K, i.e., $K = K'' = (K')'$.*

In particular, we have:

(a) $x \geq_K y$ *if and only if* $q \cdot x \geq q \cdot y$ *for each* $q \in K'$, *and*
(b) $q_1 \geq_{K'} q_2$ *if and only if* $q_1 \cdot z \geq q_2 \cdot z$ *for each* $z \in K$.

Proof. It should be clear that $K' + K' \subseteq K'$, $\alpha K' \subseteq K'$ for each $\alpha \geq 0$, and that K' is a closed subset of $\mathbb{R}^J$. To see that K' is a cone, let $q \in K' \cap (-K')$. Then, $q \cdot x \geq 0$ and $q \cdot x \leq 0$ both hold for all $x \in K$. That is, $q \cdot x = 0$ for each $x \in K$. Since K is generating, it follows that $q \cdot x = 0$ for all $x \in \mathbb{R}^J$, i.e., $q = 0$.

Clearly, $K \subseteq K''$. To see that $K = K''$ is indeed true, assume by way of contradiction that K is a proper subset of K''. So, there exists some $x \in K''$ such that $x \notin K$. Since K is closed and convex, it follows (from the separation theorem) that there exist some $q \in \mathbb{R}^J$ and some real number c such that $q \cdot y \geq c > q \cdot x$ for each $y \in K$. Since K is a cone, we get $c \leq 0$ and $q \cdot y \geq 0$ for all $y \in K$. This implies $q \in K'$, and so $q \cdot x \geq 0$, which contradicts $q \cdot x < c \leq 0$. Hence, $K = K''$.

Finally, we show that K' is generating, i.e., that $\mathbb{R}^J = K' - K'$. To see this, assume that some $q \in \mathbb{R}^J$ satisfies $q \cdot y = 0$ for all $y \in K' - K'$. This implies $q \in K'' \cap (-K'') = K \cap (-K) = \{0\}$, i.e., $q = 0$. Thus, the closed vector subspace $K' - K'$ is dense in $\mathbb{R}^J$, and consequently $\mathbb{R}^J = K' - K'$. □

A vector $q \in (\mathbb{R}^J)' = \mathbb{R}^J$ is said to be *K-strictly positive* (or simply *strictly positive*), denoted $q \gg_K 0$, if $x >_K 0$ implies $q \cdot x > 0$. The strictly positive vectors will play the role of the arbitrage free prices.

There are two more notions related to strict positivity. If K is a cone in a vector space X, then a vector $x \in K$ is said to be:

(a) *internal*, if for each $y \in X$ there exists some $\alpha_0 > 0$ such that $x + \alpha y \in K$ for all $|\alpha| \leq \alpha_0$, and
(b) *an order unit*, or simply a *unit*, if for each $y \in X$ there exists some $\alpha > 0$ such that $y \leq_K \alpha x$.

For the dual of a closed and generating cone in R^J all these notions coincide.

Lemma A1.3. *For a closed and generating cone K and some $q \in K'$ the following statements are equivalent.*

(1) *q is K-strictly positive.*
(2) *q is an interior point of K'.*
(3) *q is an internal point of K'.*
(4) *q is an order unit of K'.*

Moreover, the interior of K' is non-empty—and so the collection $(K')^\circ$ of all strictly positive vectors is dense in K'.

Proof. Notice first that from Theorem A1.2 and Lemma A1.1 we know that $(K')^\circ$ (the interior of K') is non-empty. This easily implies that $(K')^\circ$ is dense in K'.

(1) $\Longrightarrow$ (2) Let q be a strictly positive vector and assume by way of contradiction that $q \notin (K')^\circ$. Since $(K')^\circ$ in non-empty and convex, there exists (in view of the separation theorem) some non-zero vector $x \in \mathbb{R}^J$ such that $q \cdot x \leq p \cdot x$ for all $p \in (K')^\circ$. Since $(K')^\circ$ is dense in K', it follows that $q \cdot x \leq p \cdot x$ holds for all $p \in K'$. Taking into account that K' is a cone, we see that $q \cdot x \leq 0 \leq p \cdot x$ for all $p \in K'$. This implies $x \in K'' = K$, and so $x >_K 0$. But then, the strict positivity of q implies $q \cdot x > 0$, contrary to $q \cdot x \leq 0$. Thus, $q \in (K')^\circ$.

(2) $\Longrightarrow$ (3) This is obvious.

(3) $\Longrightarrow$ (4) Assume that q in internal point of K' and let $p \in \mathbb{R}^J$. Pick some $\alpha > 0$ such that $q + \alpha(-p) \in K'$. This implies $p \leq_{K'} \frac{1}{\alpha} q$, and so q is an order unit.

(4) $\Longrightarrow$ (1) Fix an interior vector p in the dual cone K'. Also, choose a symmetric neighborhood V of zero such that $p + V \subseteq K'$. From $p \pm v \in K'$ for each $v \in V$, it follows that $-p \leq_{K'} v \leq_{K'} p$ for each $v \in V$, i.e., $V \subseteq [-p,p]_{K'}$. Since q is an order unit, there exists some $\alpha > 0$ such that $\alpha q \geq_{K'} \pm p$, and hence $\frac{1}{\alpha}[-p,p]_{K'} \subseteq [-q,q]_{K'}$. So, if we let $W = \frac{1}{\alpha} V$, then $W \subseteq [-q,q]_{K'}$, and thus $q + W \subseteq [0,2p]_{K'} \subseteq K'$. This shows that q is an interior point of K'.

Now let $x >_K 0$ and assume by way of contradiction that $q \cdot x = 0$. If $r \in K'$ is arbitrary, then there exists some $\lambda > 0$ such that $\pm\lambda r \in W$. This yields $q \pm \lambda r \in K'$, and so $0 \leq (q \pm \lambda r) \cdot x = \pm\lambda r \cdot x$. This implies $r \cdot x = 0$ for all $r \in K'$, and consequently $r \cdot x = 0$ for all $r \in \mathbb{R}^J$. Therefore, $x = 0$, which is impossible. This contradiction shows that $q \cdot x > 0$, and so q is strictly positive. $\square$

Lemma A1.4. *Let K be a closed and generating cone in $\mathbb{R}^J$. If q is a strictly positive vector, then a closed subset A of K is compact if and only if the set of real numbers $q \cdot A = \{q \cdot a\colon\ a \in A\}$ is bounded.*

Proof. Let A be a closed subset of K such that $q \cdot A$ is bounded, where q is a strictly positive vector. Since (according to Lemma A1.3) q is an interior point of K', there exists an open neighborhood V of zero such that $q + V \subseteq K'$. Now let $p \in \mathbb{R}^J$ be an arbitrary vector. Choose some $\lambda > 0$ such that $\pm\frac{1}{\lambda} p \in V$, and so $q \pm \frac{1}{\lambda} p \in K'$. Therefore, $q \pm \frac{1}{\lambda} p \geq_{K'} 0$ or $-\lambda q \leq_{K'} p \leq_{K'} \lambda p$. This—and the fact that $q \cdot A$ is bounded—imply that the set $p \cdot A$ is bounded for each $p \in \mathbb{R}^J$. Consequently, A is a bounded subset of $\mathbb{R}^J$. Since A is also closed, it must be a compact set. $\square$

Corollary A1.5. *If K is a closed and generating cone in $\mathbb{R}^J$, then the K-order intervals of $\mathbb{R}^J$ are compact.*

Proof. Let $[0,x]_K = K \cap (x+K)$ be an order interval. Since K is closed, it should be obvious that $[0,x]_K$ is also closed. Now fix some vector $q \in (K')^\circ$ and note that $0 \le q \cdot y \le q \cdot x$ for each $y \in [0,x]_K$, i.e., the set $q \cdot [0,x]_K$ is bounded. By Lemma A1.4, the order interval $[0,x]_K$ is compact. □

Now let K be a cone in $\mathbb{R}^J$. The K-supremum of two points $x, y \in \mathbb{R}^J$, if it exists, will be denoted $x \vee_K y$. We shall say that K is a *lattice cone* if for any two points $x, y \in \mathbb{R}^J$ the supremum $x \vee_K y$ exits. An immediate consequence of the basic duality Theorem A1.2 is the following.

Lemma A1.6. *A closed and generating cone in $\mathbb{R}^J$ is a lattice cone if and only if its dual cone K' is likewise a lattice cone.*

A non-zero vector x in a cone K is called a *K-extremal vector* if $0 \le_K y \le_K x$ implies $y = \alpha x$ for some $\alpha \ge 0$. The half-line $L(x) = \{\alpha x\colon\ \alpha \ge 0\}$ generated by a K-extremal vector x is called a *K-extremal ray* (or simply an *extremal ray*) of K.

Lemma A1.7. *For a cone K in $\mathbb{R}^J$ we have the following.*

(1) *If K is a lattice cone, then K has (aside of scalar multiples) exactly J extremal vectors (which are necessarily linearly independent) that generate the cone K.*

(2) *If K is generated by J linearly independent vectors of K, then K is a lattice cone and (aside of scalar multiples) these linearly independent vectors are the only extremal vectors of K.*

In other words, K is a lattice cone if and only if there exist J linearly independent vectors $e_1, e_2, \ldots, e_J$ in K that generate K, i.e.,

$$K = \Big\{ \sum_{i=1}^{J} \lambda_i e_i\colon\ \lambda_i \ge 0 \text{ for all } i = 1, 2, \ldots, J \Big\}.$$

Moreover, when K is a lattice cone, the half rays $L(e_1), L(e_2), \ldots, L(e_J)$ are the only extremal rays of K and for each pair of vectors $x = \sum_{i=1}^J \lambda_i e_i$ and $x = \sum_{i=1}^J \mu_i e_i$ we have

$$x \vee_K y = \sum_{i=1}^{J} \max\{\lambda_i, \mu_i\} e_i \quad \text{and} \quad x \wedge_K y = \sum_{i=1}^{J} \min\{\lambda_i, \mu_i\} e_i .$$

Recall that a non-empty convex subset B of a cone K is said to be a *base* for K if for each non-zero $x \in K$ with $x \ne 0$ there exist a unique vector $b \in B$ and a unique scalar $\lambda > 0$ such that $x = \lambda b$. The following simple result follows easily from the definitions.

Lemma A1.8. *If B is a base for a cone K, then (aside from scalar multiples) the extremal vectors of K are precisely the extreme points of the convex set B.*

Regarding the existence of bases we have the following result of V. Klee [10]. (For a proof see [7, Theorem 3.12.8, p. 144 and Corollary 3.12.9, p. 145].)

Lemma A1.9 (Klee). *If K is a closed cone in $\mathbb{R}^J$, then:*

(a) *K has a compact base, and*
(b) *K coincides with the convex hull of its extremal vectors.*

The proof of the existence of our cheapest hedge will be based upon the following duality result that is a special case of a result in [2].

Theorem A1.10. *Let K be a closed and generating cone in $\mathbb{R}^J$. Then, for any $x, y \in \mathbb{R}^J$ and any $q \in K'$ we have*

$$\begin{aligned} \inf_{z \geq_K x,\, z \geq_K y} q \cdot z &= \max_{p \in [0,q]_{K'}} \big[p \cdot (x-y) + q \cdot y\big] \\ &= \max_{p \in [0,q]_{K'}} \big[p \cdot x + (q-p) \cdot y\big]. \end{aligned}$$

Proof. Fix $q \in K'$, and let $x, y \in \mathbb{R}^J = (R^J)''$. By Corollary A1.5, the K'-order intervals of $\mathbb{R}^J = (R^J)'$ are norm compact. Now the desired formula follows from [2, Theorem 7.6] applied to the partially ordered vector space $L = (R^J, K')$ whose order dual is $L^\sim = (R^J, K)$. □

A2. Portfolio Dominance

In this section we shall discuss the two-period securities model when there are S states and $J \leq S$ non-redundant securities. The only information needed for our analysis is the payoff matrix

$$R = \begin{bmatrix} r_1(1) & r_2(1) & \dots & r_J(1) \\ r_1(2) & r_2(2) & \dots & r_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ r_1(S) & r_2(S) & \dots & r_J(S) \end{bmatrix},$$

where $r_1, r_2, \dots, r_J$ are the J non-redundant securities. As mentioned before, the s^{th} row of the matrix R will be denoted q_s, i.e., $q_s = \big(r_1(s), r_2(s), \dots, r_J(s)\big)$.

We shall consider the matrix R as a linear operator $R\colon \Theta = \mathbb{R}^J \to \mathbb{R}^S$, where Θ is viewed as the *portfolio space* and $\mathbb{R}^S$ as the asset space. Since the rank of the matrix R is J, the matrix R as an operator from R^J to R^S is one-to-one.

The *asset span* or the *marketed space* is the range of the operator R, and is denoted M or $\langle R\rangle$. Clearly, the operator $R\colon \mathbb{R}^J \to M$ is one-to-one and surjective. We always consider the marketed space M partially ordered by the closed cone $M_+ = \mathbb{R}^S_+ \cap M$. When M_+ is lattice cone of M, then M is called a *lattice-subspace* of $\mathbb{R}^S$.

Although the non-redundant securities $r_1, r_2, \ldots, r_J$ are not assumed to be positive vectors, we shall impose the following technical condition on M.

- ASSUMPTION: *The cone M_+ is generating in M, i.e., $M = M_+ - M_+$.*

If the riskless bond is marketed, then it should be clear that M_+ is generating. Also, if each security r_i is positive, then M_+ is automatically generating. We are now ready to define the portfolio cone.

Definition A2.1. *The **portfolio cone** is the cone in the portfolio space defined by*

$$C = \{\theta \in \Theta = \mathbb{R}^J:\ R\theta \geq 0\} = \{\theta \in \mathbb{R}^J:\ q_s \cdot \theta \geq 0 \ \textit{for each}\ s = 1, 2, \ldots, J\}.$$

That is, the portfolio cone C consists of all portfolios in $\mathbb{R}^J$ with non-negative payoff and is the inverse image of the standard cone in $\mathbb{R}^S$ under the operator R, i.e., $C = R^{-1}(\mathbb{R}^S_+) = R^{-1}(M_+)$. This easily implies that C is a closed cone in Θ, and our basic assumption shows that we have following.

Lemma A2.2. *The portfolio cone C is closed and generating.*

Recall that the vectors in $\Theta' = (\mathbb{R}^J)'$ are also known as *security prices.* If $p \in \Theta'$ and $\theta \in \Theta$, then $p \cdot \theta$ represents the value of the portfolio θ ate prices p. The prices in the dual cone of C are known as weakly arbitrage prices.

Definition A2.3. *A **weakly arbitrage free price** is a price lying in the dual cone of the portfolio cone C. That is, the weakly arbitrage free prices are the prices in*

$$C' = \{q \in \Theta' = \mathbb{R}^J:\ q \cdot \theta \geq 0 \ \textit{for all}\ \theta \in C\}.$$

A price $q \in C'$ is said to be *arbitrage free* if $\theta \in C$ and $\theta \neq 0$ imply $q \cdot \theta > 0$. That is, the arbitrage free prices are the C-strictly positive vectors—which, according to Lemma A1.3, they are precisely the vectors in $(C')^{\circ}$. Since $(C')^{\circ}$ is dense in C', we have the following important property.

Lemma A2.4. *The cone of weakly arbitrage free prices C' is closed and generating and is precisely the closure of the convex set $(C')^{\circ}$ of all arbitrage free prices.*

Specializing Lemma A1.6 to C and C' we have the following.

Lemma A2.5. *The three statements below are equivalent.*

(1) *The portfolio cone C is a lattice cone.*
(2) *The cone of weakly arbitrage free prices C' is a lattice cone.*
(3) *The marketed space M is a lattice-subspace of $\mathbb{R}^S$.*

We now come to the notions of dominance by portfolios and prices.

Definition A2.6. *A portfolio θ is said to **dominate** another portfolio η if $\theta \geq_C \eta$, i.e., if $R\theta \geq R\eta$.*

*Similarly, a weakly arbitrage free price q **dominates** another weakly arbitrage free price p if $q \geq_{C'} p$, that is, if for any portfolio $\theta \in C$ we have $q \cdot \theta \geq p \cdot \theta$.*

Since $R\theta \geq 0$ is equivalent to $R\theta \cdot y \geq 0$ for all $y \in \mathbb{R}^S_+$ and $R\theta \cdot y = \theta \cdot R^t y$ holds (where R^t denotes the transpose of the matrix R), it follows that $R^t y$ belongs to C' for each $y \in \mathbb{R}^J_+$. That is, we have the inclusion $\{R^t y\colon\ y \in \mathbb{R}^J_+\} \subseteq C'$, where $\{R^t y\colon\ y \in \mathbb{R}^J_+\}$ is clearly the (closed) cone generated by the rows of the payoff matrix R. The next results informs that, in fact, we have equality.

Lemma A2.7. *The cone of weakly arbitrage free prices C' is precisely the cone generated by the rows of the payoff matrix R. That is,*

$$C' = \{R^t y\colon\ y \in \mathbb{R}^J_+\} = \Big\{\sum_{s=1}^{S} \lambda_s q_s\colon\ \lambda_s \geq 0 \ \textit{for each}\ s = 1, 2, \ldots, S\Big\}.$$

Proof. Let $C_1 = \{R^t y\colon\ y \in \mathbb{R}^J_+\}$. As noticed above, C_1 is the closed (convex) subcone of C' that is generated by the rows of the payoff matrix R. If $C_1 \neq C'$, then there exists some $q \in C'$ such that $q \notin C_1$. So, by the Separation Theorem, there exists some $\theta \in \mathbb{R}^J$ such that $r \cdot \theta \geq 0 > q \cdot \theta$ holds for all $r \in C_1$. In particular, we have $q_s \cdot \theta \geq 0$ for each s, and so $\theta \in C$. This implies $q \cdot \theta \geq 0$, which contradicts $q \cdot \theta < 0$. This contradiction establishes that $C_1 = C'$. $\square$

The next result presents a connection between the extremal rays of C' and the rows of the payoff matrix R. This is a basic result for our work.

Theorem A2.8. *The cone of weakly arbitrage free prices C' enjoys the following properties.*

(1) *Every extremal ray of C' coincides with the half ray generated by some row of R (and so C' has a finite number of extremal rays).*
(2) *The number ℓ of all extremal rays of C' satisfies $J \le \ell \le S$. In particular, C' is a lattice cone if and only if $\ell = J$.*

Proof. (1) Let q be an extremal vector of C' and let $L(q)$ be its half-ray. By Lemma A2.7, there exist row vectors $q_{s_1}, \dots, q_{s_k}$ of the payoff matrix R and positive constants $\alpha_1, \dots, \alpha_k$ such that $q = \sum_{i=1}^{k} \alpha_i q_{s_i}$. From $0 \le_{C'} \alpha_1 q_{s_1} \le_{C'} q$ and the extremality of q, there exists some $\lambda > 0$ such that $\alpha_1 q_{s_1} = \lambda q$. Hence, $q = \mu q_{s_1}$ holds for some $\mu > 0$, and so $L(q) = L(q_{s_1})$. This shows that C' has a finite number of extremal rays.

(2) Let ℓ be the number of extremal rays of C'. By part (1), it follows that $\ell \le S$. Also, let $q_{s_1}, \dots, q_{s_\ell}$ be ℓ rows of R that generate all extremal rays of C'.

By Lemma A1.9, we know that C' is the convex hull of its extremal vectors. This implies that C' is generated by the row vectors $q_{s_1}, \dots, q_{s_\ell}$. In particular, from $\mathbb{R}^J = C' - C'$ it follows that $\ell \ge J$. Otherwise, if $\ell < J$ were true, then the vector space $C' - C'$ could not be of dimension J.

For the last part, notice first that if $\ell = J$, then the vectors $q_{s_1}, \dots, q_{s_\ell}$ must be linearly independent. This implies that the cone C' must be a lattice cone. On the other hand, if C' is a lattice cone, then it must have exactly J extremal rays, in which case we infer that $\ell = J$. □

We are now ready to discuss the existence of cheapest hedging portfolios.

Theorem A2.9. *For any portfolio θ and any arbitrage free price q there exists a portfolio θ^* such that: its payoff is positive, it is dominating θ, and*

$$q \cdot \theta^* = \min_{\eta \ge_C \theta,\, \eta \ge_C 0} q \cdot \eta = \max_{0 \le_{C'} p \le_{C'} q} p \cdot \theta\,.$$

Proof. Fix a portfolio θ and let q be an arbitrage free price. Since C has interior points, there exists some $\eta_1 \in C$ such that $\eta_1 \ge_C \theta$. Now consider the convex set

$$A = \{\eta \in C\colon\ \eta \ge_C \theta \text{ and } q \cdot \eta \le q \cdot \eta_1\}\,.$$

Clearly, A is a closed subset of C and $q \cdot A$ is bounded. By Lemma A1.4, the set A is compact. Now, from $\eta_1 \in C$ and $\eta_1 \ge_C \theta$, we see that $\eta_1 \in A$. To complete the proof notice that

$$\inf_{\eta \in A} q \cdot \eta = \inf_{\eta \ge_C \theta,\, \eta \ge_C 0} q \cdot \eta\,,$$

and then use Theorem A1.10 and the compactness of A. □

Corollary A2.10. *Let θ_1 and θ_2 be two portfolios, and let q be an arbitrage free price. Then there exists a portfolio θ^* dominating θ_1 and θ_2 such that*

$$q \cdot \theta^* = \min_{\eta \geq_C \theta_1, \eta \geq_C \theta_2} q \cdot \eta = \max_{0 \leq_{C'} p \leq_{C'} q} \left[p \cdot \theta_1 + (q - p) \cdot \theta_2 \right].$$

Proof. By Theorem A2.9 there exists some portfolio ϵ such that

$$q \cdot \epsilon = \min_{\eta \geq_C \theta_1 - \theta_2, \eta \geq_C 0} q \cdot \eta = \max_{0 \leq_{C'} p \leq_{C'} q} p \cdot (\theta_1 - \theta_2).$$

Now if we let $\theta^* = \epsilon + \theta_2$, then it is easy to check that θ^* satisfies the desired properties. □

Any portfolio θ^* dominating θ_1 and θ_2 satisfying the optimality equation of Corollary A2.10 is known as a *cheapest hedging portfolio* (or a *minimum-premium insurance portfolio*) for θ_1 and θ_2 with respect to the arbitrage free price q.

In [1] it was shown that a unique minimum-premium insurance portfolio exists for any pair of portfolios that is independent of the arbitrage free price if and only if C is a lattice cone. We can prove that result easily from our analysis here.

Lemma A2.11 (Aliprantis–Brown–Werner). *The following are equivalent:*

(1) *Each pair of portfolios θ_1 and θ_2 admits a unique minimum-premium insurance portfolio θ^* that is independent of the arbitrage free price. That is, for each pair θ_1 and θ_2 of portfolios there exists a unique portfolio θ^* dominating θ_1 and θ_2 such that for each arbitrage free price q we have*

$$q \cdot \theta^* = \min_{\eta \geq_C \theta_1, \eta \geq_C \theta_2} q \cdot \eta.$$

(2) *The portfolio cone C is a lattice cone in $\mathbb{R}^J$ or, equivalently, the marketed space M is a lattice-subspace of $\mathbb{R}^S$.*

In particular, if C is a lattice cone, then the unique portfolio θ^* that satisfies property (1) is the portfolio $\theta^* = \theta_1 \vee_C \theta_2$.

Proof. (1) $\Longrightarrow$ (2) Assume that θ^* has the stated uniqueness property. If some portfolio η satisfies $\eta \geq_C \theta_1$ and $\eta \geq_C \theta_2$, then we have $q \cdot \eta \geq q \cdot \theta^*$ for each arbitrage free price q. Since the arbitrage free prices are dense in C', we see that $q \cdot \eta \geq q \cdot \theta^*$ for each $q \in C'$. By Theorem A1.2, we get $\eta \geq \theta^*$, and this shows that $\theta^* = \theta_1 \vee_C \theta_2$.

(1) $\Longrightarrow$ (2) If C is a lattice cone, then it is easy to see that the portfolio $\theta^* = \theta_1 \vee_C \theta_2$ satisfies the properties stated in (1). □

A3. The Proof of Theorem 3.2

For any non-empty subset I of the index set of states $\{1, 2, \ldots, S\}$, let H_I be the vector subspace generated in $\mathbb{R}^J$ by the collection of the row vectors $\{q_s\colon\ s \in I\}$. Clearly, there is a finite number of distinct vector subspaces of the form H_I. Let

$$\mathcal{H} = \bigcup_{\{I\colon\ \dim H_I < J\}} H_I\,.$$

Thus, the set $\mathcal{H}$ is a (finite) union of vector subspaces. As expected, the closed set $\mathcal{H}$ has an empty interior.

Lemma A3.1. *The set $\mathcal{H}$ is closed and has no interior points. In particular, the set of arbitrage free prices not in $\mathcal{H}$ is open and dense in the set of arbitrage free prices.*

Proof. Clearly, each H_I is a closed subspace of $\mathbb{R}^J$. Since $\dim H_I < J$ implies $H_I^\circ = \emptyset$, it follows that $\mathcal{H}$ is a finite union of closed sets with empty interior. The conclusion now follows from the following topological fact.

($\bullet$) *If $C_1, C_2, \ldots, C_k$ are closed subsets of a topological space such that $C_i^\circ = \emptyset$ holds for each i, then the closed set $C = \bigcup_{i=1}^k C_i$ has an empty interior.*

A proof of the preceding claim goes as follows. Assume that x is an interior point of $C = \bigcup_{i=1}^k C_i$. Pick an open neighborhood N of x such that $N \subseteq C$. Since x is not an interior point of C_1, there exists some point $x_1 \in N$ such that $x_1 \notin C_1$. Thus, x belongs to the open set C_1^c, and so there exists an open neighborhood N_1 of x_1 such that $N_1 \cap C_1 = \emptyset$. Replacing N_1 by $N \cap N_1$, we can assume that $N_1 \subseteq N$.

Similarly, since x_1 is not an interior point of C_2 there exists some point $x_2 \in N_1$ and an open neighborhood N_2 of x_2 satisfying $N_2 \subseteq N_1$ and $N_2 \cap C_2 = \emptyset$. Proceeding this way, we see that there exist points $x_1, x_2, \ldots, x_k$ and open sets $N_1, N_2, \ldots, N_k$ such that $x_i \in N_i$ and $N_i \cap C_i = \emptyset$ for each $1 \le i \le k$, and

$$N_k \subseteq N_{k-1} \subseteq \cdots \subseteq N_2 \subseteq N_1 \subseteq N \subseteq C\,.$$

Now notice that

$$\emptyset \ne N_k = N_k \cap C = N_k \cap \Big(\bigcup_{i=1}^k C_i\Big) = \bigcup_{i=1}^k N_k \cap C_i \subseteq \bigcup_{i=1}^k N_i \cap C_i = \emptyset\,,$$

which is impossible. This contradiction completes the proof of ($\bullet$).

For the last claim observe that the set of arbitrage free prices is $(C')^\circ$ satisfies

$$(C')^\circ = (C')^\circ \cap \overline{\mathcal{H}^c} \subseteq \overline{(C')^\circ \cap \mathcal{H}^c} \subseteq \overline{(C')^\circ} = C'\,.$$

Since $(C')^\circ$ is dense in C', we infer that $\overline{(C')^\circ \cap \mathcal{H}^c} = C'$. $\square$

Recall that a subset $I = \{s_1, s_2, \ldots, s_J\}$ of the set of states $\{1, 2, \ldots, S\}$ defines a *pseudo-complete market* if the $J \times J$ matrix R_I with rows the vectors $q_{s_1}, q_{s_2}, \ldots, q_{s_J}$ is invertible. In this case, we also say that R_I is a *pseudo-complete market.*

The basic result needed to prove Theorem 3.2 is the following.

Lemma A3.2. *If θ is an arbitrary portfolio and q is an arbitrage free price, then there exists a portfolio θ^* such that:*

(1) *θ^* dominates θ and has positive payoff, i.e., $\theta^* \geq_C \theta$ and $\theta^* \geq_C 0$.*

(2) *θ^* solves the optimization problem*

$$q \cdot \theta^* = \min_{\eta \geq_C \theta,\, \eta \geq_C 0} q \cdot \eta\,.$$

(3) *$\theta^* = R_I^{-1} \max\{R_I\theta, 0\}$ for some pseudo-complete market R_I.*

Proof. If $\theta \in -C$, i.e., if $\theta \leq_C 0$, then the conclusion should be obvious; the portfolio $\theta^* = 0$ does the job. So, we can suppose that $\theta \notin -C$. We shall assume first that the arbitrage free price q does not belong to $\mathcal{H}$, i.e., $q \notin \mathcal{H}$.

By Theorem A2.9 there exists a portfolio θ^* that satisfies (1) and

$$q \cdot \theta^* = \min_{\eta \geq_C \theta,\, \eta \geq_C 0} q \cdot \eta = \max_{0 \leq_{C'} p \leq_{C'} q} p \cdot \theta\,.$$

Since $\theta \notin -C$, it follows from $\theta^* \geq_C \theta$ and $\theta^* \geq_C 0$ that $\theta^* >_C 0$. Consequently, the strict positivity of q implies $q \cdot \theta^* > 0$. Under the assumption $q \notin \mathcal{H}$ we shall verify next that this θ^* also satisfies (3).

Start by observing that since the order interval of security prices $[0, q]_{C'}$ is compact, there exists some $p^* \in [0, q]_{C'}$ such that

$$p^* \cdot \theta = \max_{0 \leq_{C'} p \leq_{C'} q} p \cdot \theta\,. \tag{$\star$}$$

From $p^* \leq_{C'} q$, $\theta \leq_C \theta^*$ and (2), we get $p^* \cdot \theta^* \leq q \cdot \theta^* = p^* \cdot \theta \leq p^* \cdot \theta^*$. Therefore,

$$p^* \cdot \theta^* = p^* \cdot \theta = q \cdot \theta^* > 0\,. \tag{$\star\star$}$$

In particular, we have $p^* \neq 0$.

Since $p^* \in C'$, there exist (in view of Lemma A2.7) a non-empty set of states I_1 and positive constants $\{\alpha_s\colon\ s \in I_1\}$ such that $p^* = \sum_{s \in I_1} \alpha_s q_s$. We claim that $q_s \cdot \theta \geq 0$ holds for each $s \in I_1$. To see this, assume that for some $s_0 \in I_1$ we have $q_{s_0} \cdot \theta < 0$. From $\sum_{s \in I_1} \alpha_s (q_s \cdot \theta) = p^* \cdot \theta = q \cdot \theta^* > 0$, it follows that I_1 must have at least two states. Now notice that the inequalities

$$\Big[\sum_{s \in I_1 \setminus \{s_0\}} \alpha_s q_s\Big] \cdot \theta = \sum_{s \in I_1 \setminus \{s_0\}} \alpha_s (q_s \cdot \theta) > \sum_{s \in I_1} \alpha_s (q_s \cdot \theta) = p^* \cdot \theta\,,$$

and $0 \leq_{C'} \sum_{s \in I_1 \setminus \{s_0\}} \alpha_s q_s \leq_{C'} \sum_{s \in I_1} \alpha_s q_s = p^* \leq_{C'} q$ contradict ($\star$). So, $q_s \cdot \theta \geq 0$ for each $s \in I_1$.

From $(\star\star)$ we have $\sum_{s\in I_1} \alpha_s(q_s \cdot \theta) = \sum_{s\in I_1} \alpha_s(q_s \cdot \theta^*)$. Taking into account that $\theta \leq_C \theta^*$ is equivalent to $q_s \cdot \theta \leq q_s \cdot \theta^*$ for each $s = 1, 2, \ldots, S$, it follows that $q_s \cdot \theta = q_s \cdot \theta^* \geq 0$ for each $s \in I_1$. Therefore,

$$q_s \cdot \theta^* = \max\{q_s \cdot \theta, 0\} \quad \text{for each} \quad s \in I_1 . \tag{$\dagger$}$$

Next, notice that $p^* \in [0, q]_{C'}$ implies $q - p^* \in [0, q]_{C'}$. If $q - p^* = 0$, let $I_2 = \emptyset$. If $q - p^* >_{C'} 0$, let I_2 be a non-empty subset of $\{1, 2, \ldots, S\}$ for which there exist positive scalars $\{\beta_s\colon\ s \in I_2\}$ such that $q - p^* = \sum_{s\in I_2} \beta_s q_s$. From $(\star\star)$, it follows that $\sum_{s\in I_2} \beta_s(q_s \cdot \theta^*) = (q - p^*) \cdot \theta^* = 0$. Since $\theta^* \geq_C 0$ is equivalent to $q_s \cdot \theta^* \geq 0$ for each s, the latter implies $q_s \cdot \theta^* = 0$ for each $s \in I_2$. In particular, from $\theta \leq_C \theta^*$ we infer that $q_s \cdot \theta \leq q_s \cdot \theta^* = 0$ holds for all $s \in I_2$. This shows that

$$q_s \cdot \theta^* = \max\{q_s \cdot \theta, 0\} \quad \text{for each} \quad s \in I_2 . \tag{$\dagger\dagger$}$$

By assumption $q \notin \mathcal{H}$. So, from $q = p^* + (q - p^*) \in H_{I_1 \cup I_2}$, it follows that $\dim H_{I_1 \cup I_2} = J$. This guarantees the existence of J linearly independent row vectors in $\{q_s\colon s \in I_1 \cup I_2\}$. Let $I = \{s_1, s_2, \ldots, s_J\} \subseteq I_1 \cup I_2$ be such a set of J states for which the set of vectors $\{q_s\colon\ s \in I\}$ is linearly independent. From $(\dagger)$ and $(\dagger\dagger)$, we see that

$$R_I\theta^* = \begin{bmatrix} q_{s_1} \cdot \theta^* \\ q_{s_2} \cdot \theta^* \\ \vdots \\ q_{s_J} \cdot \theta^* \end{bmatrix} = \begin{bmatrix} \max\{q_{s_1} \cdot \theta, 0\} \\ \max\{q_{s_2} \cdot \theta, 0\} \\ \vdots \\ \max\{q_{s_J} \cdot \theta, 0\} \end{bmatrix} = \max\left\{ \begin{bmatrix} q_{s_1} \cdot \theta \\ q_{s_2} \cdot \theta \\ \vdots \\ q_{s_J} \cdot \theta \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} = \max\{R_I\theta, 0\} .$$

Finally, notice that the $J \times J$ square matrix R_I has rank J and so it is invertible. Consequently, $\theta^* = R_I^{-1} \max\{R_I\theta, 0\}$, and the validity of (3) has been established.

Next, we consider the case $q \in \mathcal{H}$. By Lemma A3.1 there exists a sequence $\{q_n\}$ of arbitrage free prices such that $q_n \to q$ and $q_n \notin \mathcal{H}$ for each n. By Theorem A2.9, for each n there exists a portfolio θ_n^* dominating θ with positive payoff satisfying

$$q_n \cdot \theta_n^* = \min_{\eta \geq_C \theta, \eta \geq_C 0} q_n \cdot \eta = \max_{0 \leq_{C'} p \leq_{C'} q_n} p \cdot \theta .$$

By the preceding case, for each n there exists a set I_n of J states such that

$$\theta_n^* = R_{I_n}^{-1} \max\{R_{I_n}\theta, 0\} .$$

Since there is only a finite number of subsets of the set of states $\{1, 2, \ldots, S\}$, we can assume (by passing to a subsequence if necessary) that there exists a fixed subset I of J indices such that $I_n = I$ for each n. This implies

$$\theta_n^* = R_I^{-1} \max\{R_I\theta, 0\} = \theta^*$$

for each n. We shall show that θ^* satisfies properties (1), (2), and (3). Clearly, (1) and (3) are satisfied automatically. So, to finish the proof, we must prove the validity of (2).

To this end, take any $\eta \geq_C 0$ satisfying $\eta \geq_C \theta$. Then, we have $q_n \cdot \eta \geq q \cdot \theta_n^* = q \cdot \theta^*$ for all n. Taking limits yields $q \cdot \eta \geq q \cdot \theta^*$. This shows that θ^* is a solution to the optimization problem

$$\min_{\eta \geq_C \theta, \eta \geq_C 0} q \cdot \eta ,$$

and the proof is finished. □

Corollary A3.3. *If θ_1 and θ_2 are arbitrary portfolios and q is an arbitrage free price, then there exists a portfolio θ^* such that:*

(1) *θ^* dominates θ_1 and θ_2.*

(2) *θ^* solves the optimization problem*

$$q \cdot \theta^* = \min_{\eta \geq_C \theta_1, \eta \geq_C \theta_2} q \cdot \eta .$$

(3) *$\theta^* = R_I^{-1} \max\{R_I\theta_1, R_I\theta_2\}$ for some pseudo-complete market R_I.*

Proof. Consider the portfolio $\theta = \theta_1 - \theta_2$. According to Lemma A3.2 there exists a portfolio ϵ^* such that:

(a) ϵ^* dominates θ and has positive payoff.

(b) ϵ^* solves the optimization problem

$$q \cdot \epsilon^* = \min_{\eta \geq_C \theta, \eta \geq_C 0} q \cdot \eta = \max_{0 \leq_{C'} p \leq_{C'} q} p \cdot \theta .$$

(c) $\epsilon^* = R_I^{-1} \max\{R_I\theta, 0\}$ for some pseudo-complete market R_I.

Now let $\theta^* = \epsilon^* + \theta_2$ and note that θ^* satisfies properties (1), (2), and (3). □

Finally, we are ready to prove the Cheapest Hedge Theorem 3.2. Start by observing that since the bond $\mathbf{k}$ is marketed, there exists some portfolio $\theta_1 \in \mathbb{R}^J$ such that $R\theta_1 = \mathbf{k}$. By Corollary A3.3 there exists some portfolio θ^* such that:

(i) θ^* dominates θ and θ_1.

(ii) θ^* solves the optimization problem

$$q \cdot \theta^* = \min_{\eta \geq_C \theta, \eta \geq_C \theta_1} q \cdot \eta .$$

(iii) $\theta^* = R_I^{-1} \max\{R_I\theta, R_I\theta_1\}$ for some pseudo-complete market R_I.

Next, consider the finite minimization problem:

$$(\mathcal{F}MP) \qquad \min\, q \cdot \eta$$
$$\text{s. t.: } \eta \in \mathcal{P}_{\theta,k}\,,$$

where $\mathcal{P}_{\theta,k}$ is the set of all potentially insuring portfolios of θ at the floor k, i.e.,

$$\mathcal{P}_{\theta,k} = \left\{\eta \in \mathbb{R}^J : \ \ \eta = \eta_I \ \text{ for some pseudo-complete market } R_I \text{ and } \ R\eta \geq R\theta \vee \mathbf{k}\right\}.$$

From (i), (ii), and (iii), we see that the portfolio θ^* is a solution of the minimization problem $(\mathcal{F}MP)$, and that any solution of $(\mathcal{F}MP)$ satisfies (i), (ii), and (iii). Now the validity of all statements in Theorem 3.2 follow from this equivalence.

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